Profinite monoids as polyadic spaces

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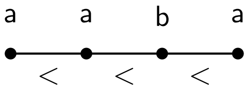
Finite words and logic

Let Σ be a finite alphabet.

The set of finite words over Σ is a free **monoid** generated by Σ.

A word $w \in \Sigma^*$ may be viewed as a finite structure with linear order $<$ and a decomposition into unary predicates $(P_a)_{a \in \Sigma}$:



Every **monadic second order sentence** in signature

$$S_\Sigma := \{<\} \cup \{P_a \ : \ a \in \Sigma\}$$

describes a set of finite Σ-words.

Regular sets

A set L of finite words is **regular** if it satisfies the following equivalent conditions:

- L is definable by a monadic second order sentence,
- L is recognizable by a finite automaton,
- L is saturated under a finite index monoid congruence on Σ^*, i.e., there exists a surjective homomorphism

  $$h\colon \Sigma^* \twoheadrightarrow M,$$

  with M a finite monoid, such that, for some $P \subseteq M$,

  $$L = h^{-1}(P).$$

The free profinite monoid

The **free profinite monoid** over Σ is the, up to isomorphism unique, embedding of Σ into a topological monoid Σ^{pro} such that, for every finite monoid M and function $f : \Sigma \to M^{\mathsf{set}}$, there exists a unique continuous homomorphism $\overline{f} : \Sigma^{\mathsf{pro}} \to M^{\mathsf{disc}}$ that extends f.

$$\begin{array}{ccc} & \mathbf{FinMon} & \\ {\scriptstyle (-)^{\mathsf{set}}} \swarrow & & \searrow {\scriptstyle (-)^{\mathsf{disc}}} \\ \mathbf{FinSet} & \xrightarrow{\quad (-)^{\mathsf{pro}} \quad} & \mathbf{TopMon} \end{array}$$

$$\frac{f : \Sigma \to M^{\mathsf{set}}}{\overline{f} : \Sigma^{\mathsf{pro}} \to M^{\mathsf{disc}}}$$

Elements of Σ^{pro} are called **profinite words** over Σ.

Characterizing first-order logic

Theorem. A set $L \subseteq \Sigma^*$ is **first-order definable**

if, and only if,

L can be recognized by an **aperiodic** finite monoid, i.e., one satisfying the profinite equation

$$x^{\omega} = x^{\omega} x.$$

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Here, for any $x \in \Sigma^{\text{pro}}$, the ω-power x^{ω} of x is defined as the unique idempotent element in the orbit-closure of x.

Aperiodicity is equivalent to the absence of non-trivial subgroups.

We get the monoid of **proaperiodic words** as the quotient

$$\Sigma^{\text{ap}} := \Sigma^{\text{pro}} / \langle x^{\omega} = x^{\omega} x \rangle.$$

Schützenberger 1965; McNaughton & Papert 1971; Reiterman 1982

Constructions of the free profinite monoid

Theorem. The topological space underlying the free profinite monoid Σ^{pro} is homeomorphic to all of the following:

- the limit in **TopMon** of a projective diagram of finite monoids,
- an ultrametric completion of Σ^*,
- the ultrafilter space of the Boolean algebra of regular sets.

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Theorem. The multiplication on Σ^{pro} is dual to a residuation structure on the regular subsets of Σ.

Reiterman 1982; Gehrke, Grigorieff & Pin 2008

Two instances

1. The equivalence

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induces a homeomorphism

$$\Sigma^{\mathsf{ap}} \cong \text{completions of the FO-theory of finite words.}$$

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2. Similarly, the equivalence

$$\text{regular} \Leftrightarrow \text{MSO-definable}$$

induces a homeomorphism

$$\Sigma^{\text{pro}} \cong \text{completions of the MSO-theory of finite words.}$$

The proaperiodic monoid of 0-types

Let Σ be a finite alphabet. Consider the FO-theory of finite words:

$$T \stackrel{\text{def}}{=} \{\varphi \text{ an FO-sentence} \quad | \quad \text{for all } w \in \Sigma^*, w \models \varphi\} \ .$$

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$$T \stackrel{\text{def}}{=} \{\varphi \text{ an FO-sentence } \mid \text{ for all } w \in \Sigma^*, w \models \varphi\} .$$

A pseudofinite word is a model of the theory T. Concretely, it is a discrete linear order with endpoints, on which the predicates $(P_a)_{a\in\Sigma}$ are a partition, satisfying an FO-induction scheme.

FO-equivalence classes of pseudofinite words are in bijection with completions or 0-types of T.

This is the Stone space $\mathcal{S}_T(0)$ of the Lindenbaum algebra of T.

The proaperiodic monoid of 0-types, continued

The space $\mathcal{S}_T(0)$ admits a continuous multiplication: concatenate.

One needs to show that the concatenation of pseudofinite words is well-defined up to FO-equivalence. (Exercise.)

Theorem

$\Sigma^{\mathsf{ap}} \cong$ *the topological monoid of pseudofinite words.*

We can use this to analyze the structure of Σ^{ap}.

Steinberg & G. 2019

Similar things work for Σ^{pro} and MSO (Linkhorn 2021).

Two questions

We have seen:

Logical Theory	Profinite Monoid of 0-types
FO	Σ^{ap}
MSO	Σ^{pro}

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We now ask:

1. When is a profinite monoid the type space of an FO theory?
2. When is the type space of an FO theory a profinite monoid?

Polyadic spaces

The dual equivalence **BoolAlg** $\simeq^{\mathrm{op}}$ **BoolSp** extends to a dual equivalence between:

Boolean hyperdoctrines and open polyadic Boolean spaces .

A polyadic Boolean space on a category **C** is a functor $\mathcal{S}\colon \mathbf{C}^{\mathrm{op}} \to$ **BoolSp** such that every span in $\int \mathcal{S}$ admits a cocone. It is open if the image of any morphism is open.

To any FO theory T we can naturally associate a type space functor $\mathcal{S}_T$ with $\mathbf{C} =$ **FinSet**.

This works much more generally, for compact ordered spaces, giving type spaces for other first-order logics.

Joyal 1971; see also e.g. G. & Marques 2024

Profinite monoids as monoidal functors

Fact. A monoid M is profinite if, and only if, M is a monoid internal to the category of Boolean (Stone) spaces.

Thus, a profinite monoid M can be encoded as a functor $\mathcal{P}_M \colon \Delta_+ \to \mathbf{BoolSp}$, where Δ_+ is the category of finite linear orders with monotone functions:

$$\mathcal{P}_M(n) \stackrel{\text{def}}{=} M^n,$$
$$\mathcal{P}_M(f \colon n \to k) \stackrel{\text{def}}{=} (x_1, \ldots, x_n) \mapsto \prod_{j \in f^{-1}(i)} x_j \ .$$

These are exactly the monoidal functors $(\Delta_+, \oplus) \to (\mathbf{BoolSp}, \times)$.

When is the monoidal functor a polyadic space?

A monoid M is equidivisible if, for any $m, n, \mu, \nu \in M$,

$$\text{if } mn = \mu\nu \text{ then there exists } x \in M \text{ such that } mx = \mu \text{ and } x\nu = n \ , \\ \text{or } \mu x = m \text{ and } xn = \nu \ .$$

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Theorem (Marquès 2021)

The monoidal functor $\mathcal{P}_M$ associated to a profinite monoid M is an open polyadic Boolean space if, and only if, the following three properties hold:

1. *M is equidivisible,*
2. *the element 1_M is isolated in the topology and the only invertible element in the monoid, and*
3. *the multiplication function $\cdot_M \colon M^2 \to M$ is an open map.*